Limit Techniques

Type of limit	$x \to$	Technique
Limits to infinity	$\pm\infty$	Divide by highest power in denominator
Fraction of polynomials	Number	Factor and cancel
$\sqrt{x+\ldots}-n$	Number	Multiply by $\frac{\sqrt{x+\ldots}+n}{\sqrt{x+\ldots}+n}$
$\frac{\sin nx}{x}$	0	This is always equal to n
$\frac{\sin}{\sin}$	0	Multiply by $\frac{x}{x}$
$\tan x$	Number	Express tan in terms of sin and cos
$1-\cos$ or $1-\sin$	$0, \frac{\pi}{2}, \pi, 2\pi$	Multiply by conjugate and use $\cos^2+\sin^2=1$

Examples:

0.1 Limits to Infinity

We need to remember 3 basic types of limits here:

$$\begin{aligned} \lim_{x\to\infty} \frac{1}{x^n} &= 0 \\ \lim_{x\to-\infty} \frac{1}{x^n} &= 0 \end{aligned} \tag{1}$$

when $n > 0$.

$$\begin{aligned} \lim_{x\to\infty} x^{\text{EVEN}} &= \infty \\ \lim_{x\to-\infty} x^{\text{EVEN}} &= \infty \end{aligned} \tag{2}$$

$$\lim_{x \to \infty} x^{\text{ODD}} = \infty$$
$$\lim_{x \to -\infty} x^{\text{ODD}} = -\infty \tag{3}$$

Example 0.1.1: •

Find $\lim_{x\to\infty} \dfrac{x^2+3x-1}{x^5+2}$

The highest power is in the denominator, so this should be zero. If we divide by the highest power in the denominator (x^5) we get

$$\lim_{x\to\infty} \frac{\frac{1}{x^3} + 3\frac{1}{x^4} - \frac{1}{x^5}}{1 + 2\frac{1}{x^5}} = \frac{0+0+0}{1+0} = 0 \tag{4}$$

Example 0.1.2: •

Find $\lim_{x\to-\infty} \dfrac{x^5-14x^4+x}{x^2+x}$

Dividing by the highest power in the denominator (x^2), we get

$$\lim_{x\to-\infty} \frac{x^3 - 14x^2 + \frac{1}{x}}{1 + \frac{1}{x}} \tag{5}$$

Now the $1/x$ terms go to 0. But $x^3 \to -\infty$. Since x^3 is the highest power left over in the numerator, it determines what happens. So

$$\lim_{x\to-\infty} \frac{x^5-14x^4+x}{x^2+x} = -\infty \tag{6}$$

0.2 Fractions of Polynomials

Example 0.2.1: •

Find $\lim_{x\to 4} \dfrac{x^2-x-12}{x-4}$

The denominator is already factored. Factoring the numerator we have

$$\lim_{x\to 4}\frac{x^2-x-12}{x-4}=\lim_{x\to 4}\frac{(x-4)(x+3)}{x-4} \tag{7}$$

Then cancelling $x-4$, we're left with

$$\lim_{x\to 4} x+3=4+3=7 \tag{8}$$

Example 0.2.2: •

Find $\lim_{x\to -3}\dfrac{x^2+5x+6}{x^2-9}$

Here we need to factor both the numerator and denominator. We get

$$\lim_{x\to -3}\frac{x^2+5x+6}{x^2-9}=\lim_{x\to -3}\frac{(x+2)(x+3)}{(x-3)(x+3)} \tag{9}$$

Cancelling $x+3$, we have

$$\lim_{x\to -3}\frac{(x+2)}{(x-3)}=\frac{-3+2}{-3-3}=\frac{-1}{-6}=1/6 \tag{10}$$

0.3 $\sqrt{x+\ldots}-n$

Example 0.3.1: •

Find $\lim_{x\to 4}\dfrac{x^2-4}{\sqrt{x}-2}$

$$\lim_{x\to 4}\frac{x^2-16}{\sqrt{x}-2}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)=\lim_{x\to 4}\frac{(x^2-16)(\sqrt{x}+2)}{x-4} \tag{11}$$

Now we can factor the numerator (because we have one polynomial divided by another), to get

$$\begin{aligned}\lim_{x\to 4}\frac{(x-4)(x+4)(\sqrt{x}+2)}{x-4} &= \lim_{x\to 4}(x+4)(\sqrt{x}+2)\\ &=(4+4)(\sqrt{4}+2)\\ &=8(2+2)\\ &=32\end{aligned} \tag{12}$$

Example 0.3.2: •

Find $\lim_{x\to 5} \dfrac{x-5}{2-\sqrt{x-1}}$

Here the conjugate is $2+\sqrt{x-1}$. Multiplying by it on the top and bottom we have

$$\lim_{x\to 5} \frac{(x-5)(2+\sqrt{x-1})}{(2-\sqrt{x-1})(2+\sqrt{x-1})} = \lim_{x\to 5} \frac{(x-5)(2+\sqrt{x-1})}{4-(x-1)} \tag{13}$$

This is

$$\lim_{x\to 5} \frac{(x-5)(2+\sqrt{x-1})}{5-x} = \lim_{x\to 5} \frac{(x-5)(2+\sqrt{x-1})}{-(x-5)} \tag{14}$$

Now we cancel the $x-5$ to get

$$\lim_{x\to 5} -(2+\sqrt{x-1}) = -(2+\sqrt{5-1}) = -(2+\sqrt{4}) = -(2+2) = -4 \tag{15}$$

0.4 $\dfrac{\sin nx}{x}$

This is a type of limit you just need to memorize:

$$\lim_{x\to 0} \frac{\sin nx}{x} = n \tag{16}$$

Example 0.4.1: •

$$\lim_{x\to 0} \frac{\sin 7x}{x} = 7 \tag{17}$$

Example 0.4.2: •

$$\lim_{x\to 0} \frac{\sin -5x}{x} = -5 \tag{18}$$

Example 0.4.3: •

$$\lim_{x\to 0} \frac{x}{\sin \pi x} = \lim_{x\to 0} \frac{1}{(\frac{\sin \pi x}{x})} = \frac{1}{\pi} \tag{19}$$

0.5 $\frac{\sin}{\sin}$

Example 0.5.1: •

Find $\lim_{x\to 0} \dfrac{\sin x}{\sin 10x}$

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x}{\sin 10x} &= \lim_{x\to 0} \frac{\sin x}{\sin 10x}(\frac{x}{x}) \\ &= \lim_{x\to 0} \frac{\sin x}{x}\frac{x}{\sin 10x} \\ &= 1(\frac{1}{10}) \\ &= \frac{1}{10}\end{aligned} \tag{20}$$

Example 0.5.2: •

Find $\lim_{x\to 0} \dfrac{\sin(-17x)}{\sin\frac{x}{2}}$

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(-17x)}{\sin\frac{x}{2}} &= \lim_{x\to 0} \frac{\sin(-17x)}{\sin\frac{x}{2}}(\frac{x}{x}) \\ &= \lim_{x\to 0} \frac{\sin(-17x)}{x}\frac{x}{\sin\frac{x}{2}} \\ &= -17(\frac{1}{1/2}) \\ &= -17(2) \\ &= -34\end{aligned} \tag{21}$$

0.6 $\tan x$

Example 0.6.1: •

Find $\lim_{x\to 0} \dfrac{\tan 15x}{\sin 8x}$

First we change the tan into sin and cos

$$\lim_{x\to 0}\frac{\tan 15x}{\sin 8x} = \lim_{x\to 0}\frac{\sin 15x}{\cos 15x}\left(\frac{1}{\sin 8x}\right) \tag{22}$$

Next since we have sin / sin we multiply by x/x:

$$\lim_{x\to 0}\frac{\sin 15x}{\cos 15x}\left(\frac{1}{\sin 8x}\right)\left(\frac{x}{x}\right) = \lim_{x\to 0}\frac{\sin 15x}{x}\left(\frac{x}{\sin 8x}\right)\left(\frac{1}{\cos 15x}\right) \tag{23}$$

Now $\cos(0) = 1$, so

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 15x}{x}\left(\frac{x}{\sin 8x}\right)\left(\frac{1}{\cos 15x}\right) &= 15\left(\frac{1}{8}\right)\left(\frac{1}{1}\right)\\ &= \frac{15}{8}\end{aligned} \tag{24}$$

0.7 $1-\cos$ or $1-\sin$

Example 0.7.1: •

Find $\lim_{x\to\pi}\dfrac{\sin^2 x}{1-\cos x}$

$$\lim_{x\to 2\pi}\frac{\sin^2 x}{1-\cos x} = \lim_{x\to 2\pi}\frac{\sin^2 x}{1-\cos x}\left(\frac{1+\cos x}{1+\cos x}\right) \tag{25}$$

$$= \lim_{x\to 2\pi}\frac{(\sin^2 x)(1+\cos x)}{1-\cos^2 x} = \lim_{x\to 2\pi}\frac{(\sin^2 x)(1+\cos x)}{\sin^2 x} \tag{26}$$

$$= \lim_{x\to 2\pi} 1+\cos x = 1+1 = 2 \tag{27}$$